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Fixed point theorems for multifunctions having KKM property on almost convex sets

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Abstract

In this paper, for two nonempty subsets X and Y of a linear space E, we define the class $\mathrm{KKM}(X, Y)$ and investigate the fixed point problem for $T \in \mathrm{KKM}(X, X)$ with X an almost convex subset of a locally convex space. Our fixed point theorem contains Lassonde fixed point theorem for Kakutani factorizable multifunctions as special case.

Keywords: Fixed point; KKM property; Almost convex

1. Introduction

For two topological spaces X and Y, an upper semi-continuous multifunction $T : X \multimap Y$ is said to be a Kakutani multifunction if either T is single-valued (in which case, Y is simply assumed to be a topological space), or $T(x)$ is a compact and convex subset of Y for any x in X (in which case, Y is assumed to be a subset of topological vector space). The

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Himmelberg fixed point theorem [6], a generalization of the famous Fan–Glicksberg fixed point theorem [4,5], says that a compact Kakutani multifunction $T : X \multimap X$ on a nonempty convex subset of a locally convex space E has a fixed point. This result was extended by Lassonde [7] to multifunctions factorizable by Kakutani multifunctions through convex sets in locally convex spaces.

The main purpose of this paper is to extend the Lassonde theorem to almost convex subsets instead of convex subsets in locally convex spaces. In Section 3, for two nonempty subsets X and Y of a topological vector space E, we start with the concept of generalized KKM mapping to introduce the class $\mathrm{KKM}(X, Y)$, and then show that for any nonempty subset X of a topological vector space E, the class $\mathcal{K}_c(X, E)$ of Kakutani factorizable multifunctions is a subclass of $\mathrm{KKM}(X, E)$. The fixed point problem for multifunctions in $\mathrm{KKM}(X, X)$ is investigated in Section 4, where we established a new fixed point theorem by showing that for an almost convex subset of a locally convex space E, any compact and closed multifunction T in $\mathrm{KKM}(X, X)$ has a fixed point. This result contains Lassonde's fixed point theorem as special case. As applications of our new fixed point theorem, some results related to von Neumann's intersection theorem is derived in Section 5.

2. Preliminaries

For a nonempty set Y, 2^Y denotes the class of all subsets of Y and $\langle Y\rangle$ denotes the class of all nonempty finite subsets of Y. A multifunction $T : X \to 2^Y$ is a function from a set X into the power set 2^Y of Y. The notation $T : X \multimap Y$ stands for a multifunction $T : X \to 2^Y$ having nonempty values.

In the sequel, for $n \geqslant 0$, Δ_n denotes the standard n-simplex of $\mathbb{R}^{n+1}$, that is,

$$\Delta_n = \left\{\alpha = (\alpha_0, \dots, \alpha_n) \in \mathbb{R}^{n+1}\colon\ \alpha_i \geqslant 0 \text{ for all } i \text{ and } \sum_{i=0}^{n} \alpha_i = 1\right\};$$

and $\{e_0, \dots, e_n\}$ is the set of the vertices of Δ_n. $\mathbb{Z}_{n+1}$ denotes the set $\{0, 1, \dots, n\}$ with addition modulo $n + 1$.

If X and Y are two subsets of a linear space E, a multifunction $F : X \multimap Y$ satisfying $\mathrm{co}(A) \subseteq F(A)$ for any $A \in \langle X\rangle$ is called a KKM mapping, where $\mathrm{co}(A)$ denotes the convex hull of A. The most important result for KKM mapping is the KKM Lemma published in 1929 due to Knaster, Kuratowski and Mazurkiewicz.

Lemma 2.1. (KKM Lemma, cf. [1,8]) *Suppose $F_0, \dots, F_n$ are closed subsets of the standard n-simplex Δ_n in $\mathbb{R}^{n+1}$. If for any nonempty subset I of $\{0, \dots, n\}$, $\mathrm{co}\{e_i\colon i \in I\} \subseteq \bigcup_{i\in I} F_i$, then $\bigcap_{i=0}^{n} F_i \neq \emptyset$.*

For a multifunction $T : X \to 2^Y$, $A \subseteq X$ and $B \subseteq Y$, the image of A under T is the set $T(A) = \bigcup_{x\in A} T(x)$; and the inverse image of B under T is $T^-(B) = \{x \in X\colon\ T(x) \cap B \neq \emptyset\}$.

All topological spaces of concern are supposed to be Hausdorff. The closure of a subset X of a topological space is denoted by $\overline{X}$. Let X and Y be two topological spaces. A multifunction $T : X \to 2^Y$ is said to be

(a) upper semicontinuous (u.s.c.) if $T^{-}(B)$ is closed in X for each closed subset B of Y;
(b) lower semicontinuous (l.s.c.) if $T^{-}(B)$ is open in X for each open subset B of Y;
(c) compact if $T(X)$ is contained in a compact subset of Y;
(d) closed if its graph $\mathrm{Gr}(T) = \{(x, y)\colon\ y \in T(x),\ x \in X\}$ is a closed subset of $X \times Y$.

Lemma 2.2. (Lassonde [7, Proposition 1]) *Let X, Y, Z and X_i, Y_i $(i = 1, 2)$ be topological spaces and $T : X \multimap Y$.*

(a) *If Y is regular and T is u.s.c. with closed values, then T is closed.*
(b) *If $T : X \multimap Y$ is u.s.c. with compact values and $S : X \multimap Y$ is closed, then we have $T \cap S : X \multimap Y$, defined by $(T \cap S)(x) = T(x) \cap S(x)$ for each $x \in X$, is u.s.c.; in particular, if S is compact and closed, then S is u.s.c.*
(c) *If T is u.s.c. and compact-valued, then $T(A)$ is compact for any compact subset A of X.*
(d) *If $T : X \multimap Y$ and $S : Y \multimap Z$ are u.s.c., then the composition $ST : X \multimap Z$, defined by $S(T(x)) = \bigcup_{y \in T(x)} S(y)$ for each $x \in X$, is u.s.c.*
(e) *If $T_i : X_i \multimap Y_i$ $(i = 1, 2)$, are u.s.c., then*

$$T_1 \times T_2 : X_1 \times X_2 \multimap Y_1 \times Y_2$$

defined by $(T_1 \times T_2)(x_1, x_2) = T_1(x_1) \times T_2(x_2)$ for $(x_1, x_2) \in X_1 \times X_2$, is u.s.c.

$\mathcal{C}(X, Y)$ denotes the class of all continuous (single-valued) functions from X to Y.

For any nonempty subset X of a linear space E, a function $\varphi : X \to \mathbb{R}$ is said to be quasi-convex (respectively quasi-concave) if for any $x, y \in X$ and for any $z \in \mathrm{co}\{x, y\} \cap X$, $\varphi(z) \leqslant \max\{\varphi(x), \varphi(y)\}$ (respectively $\varphi(z) \geqslant \min\{\varphi(x), \varphi(y)\}$).

3. The class KKM(X, E)

Motivated by the works of Chang and Zhang [2], we make the following definition.

Definition 3.1. Let X and Y be two nonempty subsets of a linear space E. If $F : X \multimap Y$ satisfies that for any $\{x_1, \dots, x_n\} \in \langle X\rangle$, there is $\{y_1, \dots, y_n\} \in \langle X\rangle$ such that

$$\mathrm{co}\{y_i\colon\ i \in I\} \subseteq \bigcup_{i \in I} F(x_i)$$

for any nonempty subset I of $\{1, \dots, n\}$, then F is called a generalized KKM mapping.

It is easy to see that a KKM mapping is a generalized KKM mapping by putting $y_i = x_i$ $(i = 1, \dots, n)$ for any $\{x_1, \dots, x_n\} \in \langle X\rangle$. However, a generalized KKM mapping may not be a KKM mapping as the following example shows.

Example 3.2. Suppose $\varphi : X \to \mathbb{R}$ is any real-valued function and define $F : X \multimap X$ by $F(x) = \{y \in X\colon\ \varphi(y) \leqslant \varphi(x)\}$. Then

(a) F is always a generalized KKM mapping;
(b) If F is a KKM mapping, then φ is quasi-convex. Consequently, F is not a KKM mapping provided that φ is not quasi-convex.

In fact, for any $\{x_1, \ldots, x_n\} \in \langle X\rangle$, choosing $y \in X$ so that

$$\varphi(y) = \min\bigl\{\varphi(x_1), \ldots, \varphi(x_n)\bigr\}$$

and putting $y_i = y$ for $i = 1, \ldots, n$, we see that $\mathrm{co}\{y_i\colon i \in I\} \subseteq \bigcup_{i\in I} F(x_i)$ for any nonempty subset I of $\{1, \ldots, n\}$, and so F is a generalized KKM mapping. Next, suppose F is a KKM mapping and $z \in \mathrm{co}\{x, y\} \cap X$. Then, it follows from $z \in \mathrm{co}\{x, y\} \subseteq F(x) \cup F(y)$ that z belongs to at least one of $F(x)$ and $F(y)$, so $\varphi(z) \leqslant \max\{\varphi(x), \varphi(y)\}$, which shows that φ is quasi-convex.

Lemma 3.3. *Let X be a nonempty subset of a topological vector space E. If $F : X \multimap E$ is a generalized KKM mapping, then $\{\overline{F(x)}\colon x \in X\}$ has the finite intersection property.*

Proof. For any $\{x_0, x_1, \ldots, x_n\} \in \langle X\rangle$, since F is a generalized KKM mapping, there is $\{y_0, y_1, \ldots, y_n\} \in \langle X\rangle$ such that

$$\mathrm{co}\{y_i\colon i \in I\} \subseteq \bigcup_{i\in I} F(x_i)$$

for any nonempty subset I of $\{0, 1, \ldots, n\}$. Let $B = \mathrm{co}\{y_i\colon i = 0, 1, \ldots, n\}$ and define $G_i = \overline{F(x_i)} \cap B$ for any $i = 0, 1, \ldots, n$. Obviously, each G_i is a closed subset of B. Function $\varphi : \Delta_n \to B$ defined by $\varphi(\alpha) = \sum_{i=0}^n \alpha_i y_i$ is continuous and satisfies that

$$\varphi\bigl(\mathrm{co}\{e_i\colon i \in I\}\bigr) \subseteq \mathrm{co}\{y_i\colon i \in I\} \subseteq \bigcup_{i\in I} G_i$$

for any nonempty subset I of $\{0, 1, \ldots, n\}$. Therefore, for any $i = 0, 1, \ldots, n$, $\varphi^{-1}(G_i)$ is closed and $\mathrm{co}\{e_i\colon i \in I\} \subseteq \bigcup_{i\in I} \varphi^{-1}(G_i)$, so it follows from the KKM Lemma that $\bigcap_{i=0}^n \varphi^{-1}(G_i) \neq \emptyset$, and hence $\bigcap_{i=0}^n \varphi^{-1}(\overline{F(x_i)}) \neq \emptyset$. Any $z \in \bigcap_{i=0}^n \varphi^{-1}(\overline{F(x_i)})$ satisfies that $\varphi(z) \in \bigcap_{i=0}^n \overline{F(x_i)}$. This completes the proof. □

Example 3.4. In $\mathbb{R}^2$, let $X = [-1, 1] \times \{0\} \cup \{(0, 1)\}$ and define $F : X \multimap \mathbb{R}^2$ by $F(a, b) = \{(z, 0)\colon z \in a + [-1, 1]\}$. Then F has the finite intersection property although it is not a KKM mapping.

In fact, since $(0, 1) \notin F(0, 1) = [-1, 1] \times \{0\}$, F is not a KKM mapping. However, for any $\{x_1, \ldots, x_n\} \in \langle X\rangle$, letting

$$\begin{cases} y_i = x_i, & \text{if } x_i \in [-1, 1] \times \{0\}, \\ y_i = (0, 0), & \text{if } x_i = (0, 1), \end{cases}$$

we see that $\mathrm{co}\{y_i\colon i = 1, \ldots, n\} \subseteq [-1, 1] \times \{0\} \subseteq X$, so $\mathrm{co}\{y_i\colon i \in I\} \subseteq \bigcup_{i\in I} F(x_i)$ for any nonempty subset I of $\{1, \ldots, n\}$, which shows that F is a generalized KKM mapping. And then it follows from Lemma 3.3 that F has the finite intersection property. As a matter of fact, $(0, 0) \in F(a, b)$ for any $x = (a, b) \in X$.

Theorem 3.5. *Suppose X is a nonempty subset of a topological vector space E and $F : X \multimap X$ is a multifunction with closed values such that $F(x_0)$ is compact for some $x_0 \in X$. Then F is a generalized KKM mapping if and only if $\bigcap_{x \in X} F(x) \neq \emptyset$.*

Proof. If F is a generalized KKM mapping, then, in view of $F(x_0)$ being compact and Lemma 3.3, $\bigcap_{x \in X} F(x) \neq \emptyset$. Conversely, if $\bigcap_{x \in X} F(x) \neq \emptyset$, then $\{F(x)\colon x \in X\}$ has the finite intersection property. So for any $\{x_1, \ldots, x_n\} \in \langle X \rangle$, there is $y \in \bigcap_{i=1}^n F(x_i)$. Putting $y_i = y$ for any $i = 1, \ldots, n$, we conclude that $\operatorname{co}\{y_i\colon i \in I\} = \{y\} \subseteq \bigcup_{i \in I} F(x_i)$, for any nonempty subset I of $\{1, \ldots, n\}$. Hence F is a generalized KKM mapping. □

Similar to [3] we now extend the concept of generalized KKM mapping in the following manner.

Definition 3.6. Suppose X and Y are two nonempty subsets of a linear space E, and T, $F : X \multimap Y$. We say that F is a generalized KKM mapping with respect to T if for any $A = \{x_1, \ldots, x_n\} \in \langle X \rangle$ there is $B = \{y_1, \ldots, y_n\} \in \langle X \rangle$ satisfying

(a) $\operatorname{co}(B) \subseteq X$, and
(b) $T(\operatorname{co}\{y_i\colon i \in I\}) \subseteq \bigcup_{i \in I} F(x_i)$ for any nonempty subset I of $\{1, \ldots, n\}$.

Example 3.7. Let $X = [-2, -1] \cup [1, 2]$. Define $g : X \to X$ by $g(x) = -x$ and $F : X \multimap \mathbb{R}$ by

$$F(x) = \begin{cases} -x + [-1, 1], & \text{if } x \in [1, 2], \\ x + [-1, 1], & \text{if } x \in [-2, -1]. \end{cases}$$

For any $\{x_1, \ldots, x_n\} \in \langle X \rangle$, putting

$$y_i = \begin{cases} x_i, & \text{if } x_i \in [1, 2], \\ -x_i, & \text{if } x_i \in [-2, -1], \end{cases}$$

we see that $\operatorname{co}\{y_1, \ldots, y_n\} \subseteq [1, 2]$ and $g(\operatorname{co}\{y_i\colon i \in I\}) \subseteq [-2, -1] \subseteq \bigcup_{i \in I} F(x_i)$ for any nonempty subset I of $\{1, \ldots, n\}$. So F is a generalized KKM mapping with respect to g. Obviously, $\{F(x)\colon x \in X\}$ has the finite intersection property.

The fact that $\{F(x)\colon x \in X\}$ has the finite intersection property in the above example is generally true.

Lemma 3.8. *Suppose X and Y are two nonempty subsets of a topological vector space E and $g : X \to Y$ is continuous. If $F : X \multimap Y$ is a generalized KKM mapping with respect to g, then $\{\overline{F(x)}\colon x \in X\}$ has the finite intersection property.*

Proof. For any $\{x_1, \ldots, x_n\} \in \langle X \rangle$, choose $\{y_1, \ldots, y_n\} \in \langle X \rangle$ such that $\operatorname{co}\{y_i\colon i = 1, \ldots, n\} \subseteq X$ and $g(\operatorname{co}\{y_i\colon i \in I\}) \subseteq \bigcup_{i \in I} F(x_i)$ for any nonempty subset of $\{1, \ldots, n\}$. Then $\operatorname{co}\{y_i\colon i \in I\} \subseteq \bigcup_{i \in I} g^{-1}(\overline{F(x_i)})$ for any nonempty subset of $\{1, \ldots, n\}$. Therefore, $\bigcap_{i=1}^n g^{-1}(\overline{F(x_i)}) \neq \emptyset$ by Lemma 3.3. Any $x_0 \in \bigcap_{i=1}^n g^{-1}(\overline{F(x_i)}) \neq \emptyset$ implies that $g(x_0) \in \bigcap_{i=1}^n \overline{F(x_i)}$. □

Definition 3.9. Let X and Y be two nonempty subsets of a topological vector space E. If a multifunction $T : X \multimap Y$ satisfies that for any generalized KKM mapping $F : X \multimap Y$ with respect to T, the family $\{\overline{F(x)}\colon x \in X\}$ has the finite intersection property, then T is said to have the KKM property. The class $\mathrm{KKM}(X, Y)$ is defined to be the set

$$\{T : X \multimap Y : T \text{ has the KKM property}\}.$$

Let $\mathrm{Q}(X, Y) = \{T : X \multimap Y : T \text{ has a continuous selection}\}$. Then

$$\mathrm{Q}(X, Y) \subseteq \mathrm{KKM}(X, Y).$$

To see this, suppose $T : X \multimap Y \in \mathrm{Q}(X, Y)$ and $F : X \multimap Y$ is any generalized KKM mapping with respect to T. Let t be a continuous selection of T. It is easy to see that F is a generalized KKM mapping with respect to t, and so by Lemma 3.8 we conclude that the family $\{\overline{F(x)}\colon x \in X\}$ has the finite intersection property. Thus $\mathrm{Q}(X, Y) \subseteq \mathrm{KKM}(X, Y)$.

Let $\mathcal{K}$ be the class of Kakutani multifunctions and $\mathcal{K}_c$ the class of finite composites of multifunctions in $\mathcal{K}$, that is, $T \in \mathcal{K}_c$ if there exist two topological spaces X and Y such that

(a) $T : X \multimap Y$;
(b) $T = T_n T_{n-1} \ldots T_0$, where $T_i \in \mathcal{K}$ for $i = 0, \ldots, n$.

Lemma 3.10. (Lassonde [7, Proposition 2]) *Let X be a compact space, Y a convex subset of a topological vector space E and $T \in \mathcal{K}_c(X, Y)$. Then there is a paracompact convex subset Y' of Y and a multifunction $T' \in \mathcal{K}_c(X, Y')$ such that $T'(x) = T(x)$ for each $x \in X$.*

By using the above lemma, we proceed now to prove the following main result of this section.

Proposition 3.11. *Suppose X is a nonempty subset of a topological vector space E and $T \in \mathcal{K}_c(X, E)$. Then $T \in \mathrm{KKM}(X, E)$.*

Proof. Assume that $T \notin \mathrm{KKM}(X, E)$. Then there is a closed-valued generalized KKM mapping $F : X \multimap E$ with respect to T such that $\bigcap_{i=1}^n F(x_i) = \emptyset$ for some $\{x_1, \ldots, x_n\} \in \langle X\rangle$. Choose $\{y_1, \ldots, y_n\} \in \langle X\rangle$ such that

$$B = \mathrm{co}\{y_1, \ldots, y_n\} \subseteq X \quad \text{and} \quad T\big(\mathrm{co}\{y_i\colon i \in I\}\big) \subseteq \bigcup_{i\in I} F(x_i)$$

for any nonempty subset I of $\{1, \ldots, n\}$. We have $T|_B \in \mathcal{K}_c(B, E)$. By Lemma 3.10, there is a paracompact convex subset Y' of E and $T' \in \mathcal{K}_c(B, Y')$ such that $T'(x) = T|_B(x)$ for each $x \in B$. Define $F' : B \to 2^{Y'}$ by $F'(x) = F|_B(x) \cap Y'$ for each $x \in B$. Since $E = \bigcup_{i=1}^n F(x_i)^c$, we have that

$$Y' = E \cap Y' = \left(\bigcup_{i=1}^n F(x_i)^c\right) \cap Y' = \bigcup_{i=1}^n \big(Y' \cap F(x_i)^c\big) = \bigcup_{i=1}^n \big(Y' \setminus F(x_i)\big).$$

By the paracompactness of Y', there is a partition of unity $\{\alpha_i\}_{i=1}^n$ subordinated to $\{Y' \setminus F(x_i)\}_{i=1}^n$. Define $f : Y' \to B$ by $f(y) = \sum_{i=1}^n \alpha_i(y) y_i$ for each $y \in Y'$. Obviously, f is continuous and $f \circ T \in \mathcal{K}_c(B, B)$. It follows from Lassonde [7, Theorem 4] that $f \circ T'$ has a fixed point $\hat{x}$. Choose $\hat{y} \in T(\hat{x})$ such that $\hat{x} = f(\hat{y})$ and put $I(\hat{y}) = \{i \in \{1, \dots, n\}\colon \alpha_i(\hat{y}) > 0\}$. It is easy to see that $i \in I(\hat{y})$ if and only if $\hat{y} \notin F'(x_i)$. So $\hat{y} \notin \bigcup_{i \in I(\hat{y})} F'(x_i)$, which in view of

$$\hat{y} \in T'(\hat{x}) = T'\big(f(\hat{y})\big) \subseteq T'\big(\mathrm{co}\big\{y_i\colon i \in I(\hat{y})\big\}\big)$$

implies that $T'(\mathrm{co}\{y_i\colon i \in I(\hat{y})\}) \nsubseteq \bigcup_{i \in I(\hat{y})} F'(x_i)$. Since

$$T\big(\mathrm{co}\big\{y_i\colon i \in I(\hat{y})\big\}\big) = T'\big(\mathrm{co}\big\{y_i\colon i \in I(\hat{y})\big\}\big) \quad \text{and}$$
$$\bigcup_{i \in I(\hat{y})} F'(x_i) = \Bigg(\bigcup_{i \in I(\hat{y})} F(x_i)\Bigg) \cap Y',$$

we obtain that

$$T\big(\mathrm{co}\big\{y_i\colon i \in I(\hat{y})\big\}\big) \nsubseteq \bigcup_{i \in I(\hat{y})} F(x_i),$$

a contradiction to the fact that F is a generalized KKM mapping with respect to T. Hence, $T \in \mathrm{KKM}(X, E)$, completing the proof. □

Here, we like to mention that $\mathcal{K}_c(X, E)$ is a proper subclass of $\mathrm{KKM}(X, E)$, as the following example shows.

Example 3.12. For a nonempty subset X of a topological vector space E, define $T : X \multimap E$ by $T(x) = E \setminus \{x\}$ for any $x \in X$. Since T is not u.s.c., $T \notin \mathcal{K}_c(X, E)$. However, suppose F is any generalized KKM mapping with respect to T. Then for any $\{x_1, \dots, x_n\} \in \langle X\rangle$, there is $\{y_1, \dots, y_n\} \in \langle X\rangle$ such that $\mathrm{co}\{y_1, \dots, y_n\} \subseteq X$ and $T(\mathrm{co}\{y_i\colon i \in I\}) \subseteq \bigcup_{i \in I} F(x_i)$ for any nonempty subset I of $\{1, \dots, n\}$. In particular, $E \setminus \{y_i\} = T(y_i) \subseteq F(x_i)$ for any $i = 1, \dots, n$, so $\emptyset \neq E \setminus \{y_1, \dots, y_n\} \subseteq \bigcap_{i=1}^{n} F(x_i)$, which shows that $T \in \mathrm{KKM}(X, E)$.

Since every member T of $\mathcal{K}_c(X, X)$ may be regarded as a member of $\mathcal{K}_c(X, E)$, the following corollary comes easily.

Corollary 3.13. *Suppose X is a nonempty subset of a topological vector space E. Then $\mathcal{K}_c(X, X) \subseteq \mathrm{KKM}(X, X)$.*

4. Fixed point theorems

Himmelberg [6] introduced the concept of almost convex sets and generalized the famous Fan–Glicksberg fixed point theorem.

Definition 4.1. (Himmelberg [6]) A nonempty subset X of a topological vector space E is said to be almost convex if for any neighborhood V of 0 in E and for any $\{x_1, \dots, x_n\} \in \langle X\rangle$ there is $\{y_1, \dots, y_n\} \in \langle X\rangle$ such that $y_i - x_i \in V$ for each $i \in \{1, \dots, n\}$ and $\mathrm{co}\{y_1, \dots, y_n\} \subseteq X$.

The closure $\overline{X}$ of an almost convex subset X of a locally convex topological vector space is convex. To see this, let x, y be any two points of $\overline{X}$ and V be any symmetric convex neighborhood of the origin. Choose $x' \in X \cap (x + \frac{1}{2}V)$ and $y' \in X \cap (y + \frac{1}{2}V)$. Since X is almost convex, there are $x'', y'' \in X$ such that $x' \in x'' + \frac{1}{2}V$ and $y' \in y'' + \frac{1}{2}V$ and $\operatorname{co}\{x'', y''\} \subseteq X$. So, $x \in x'' + V$ and $y \in y'' + V$ and $\lambda x + (1-\lambda)y \in \lambda x'' + (1-\lambda)y'' + V \subseteq X + V$. Hence, $\lambda x + (1-\lambda)y \in \overline{X}$.

The following simple result will be used in Section 5.

Proposition 4.2. *Let X and Y be two almost convex subsets of topological vector spaces E and H, respectively. Then $X \times Y$ is almost convex in $E \times H$.*

Proof. Let $\{(x_1, y_1), \ldots, (x_n, y_n)\}$ be any finite subset of $X \times Y$ and U and V be any neighborhoods of the origins of E and H, respectively. Since both of X and Y are almost convex, there exist finite subsets $\{x'_1, \ldots, x'_n\}$ and $\{y'_1, \ldots, y'_n\}$ of X and Y, respectively such that

$$x'_i - x_i \in U, \quad y'_i - y_i \in V \quad \text{for each } i = 1, \ldots, n, \quad \text{and}$$
$$\operatorname{co}\{x'_i\colon i = 1, \ldots, n\} \subseteq X, \qquad \operatorname{co}\{y'_i\colon i = 1, \ldots, n\} \subseteq Y.$$

Therefore,

$$(x'_i, y'_i) - (x_i, y_i) \in U \times V \quad \text{for each } i = 1, \ldots, n,$$

and

$$\operatorname{co}\big\{(x'_i, y'_i)\colon i = 1, \ldots, n\big\} \subseteq X \times Y,$$

so $X \times Y$ is almost convex in $E \times H$. $\square$

We now devoted to the fixed point problem on an almost convex subset of a locally convex topological vector space. At first, we establish a key lemma.

Lemma 4.3. *Let X be an almost convex subset of a topological vector space E. If $T \in \mathrm{KKM}(X, X)$ is compact, then for any convex neighborhood U of 0, there is $x_U \in X$ such that $(x_U + U) \cap T(x_U) \neq \emptyset$.*

Proof. On the contrary, assume there is a convex neighborhood U of 0 such that

$$(x + U) \cap T(x) = \emptyset \tag{1}$$

for any $x \in U$. Let $K = \overline{T(X)}$. By assumption, K is compact. Define $F: X \to 2^X$ by $F(x) = K \setminus (x + \frac{1}{2}U)$ for each $x \in X$. Since $(x + U) \cap T(x) = \emptyset$, we have that

$$\emptyset \neq T(x) \subseteq K \setminus (x + U) \subseteq K \bigg\backslash \left(x + \frac{1}{2}U\right) = F(x),$$

so $F(x) \neq \emptyset$ for each $x \in X$, that is $F: X \multimap X$. We now show that F is a generalized KKM mapping with respect to T. If not, there exists $A = \{x_1, \ldots, x_n\} \in \langle X \rangle$ such that for any $B = \{y_1, \ldots, y_n\} \in \langle X \rangle$ with $\operatorname{co}(B) \subseteq X$, one has $T(\operatorname{co}\{y_i\colon i \in I\}) \nsubseteq \bigcup_{i \in I} F(x_i)$ for some nonempty subset I of $\{1,\dots,n\}$. Since X is almost convex, there is $\{z_1,\dots,z_n\}\in\langle X\rangle$ such that $\mathrm{co}\{z_1,\dots,z_n\}\subseteq X$ and

$$x_i - z_i \in \frac{1}{2}U \tag{2}$$

for any $i=1,\dots,n$. Choose a nonempty subset I of $\{1,\dots,n\}$ such that

$$T\big(\mathrm{co}\{z_i\colon i\in I\}\big)\nsubseteq \bigcup_{i\in I} F(x_i),$$

and then choose $\mu\in\mathrm{co}\{z_i\colon i\in I\}$ and $\zeta\in T(\mu)$ so that $\zeta\notin\bigcup_{i\in I}F(x_i)$. Then $\zeta\in x_i+\frac{1}{2}U$ for any $i\in I$, and so, in view of (2), $\zeta\in z_i+U$ for any $i\in I$, which implies that $\zeta\in p+U$ for any $p\in\mathrm{co}\{z_i\colon i\in I\}$. In particular, $\zeta\in\mu+U$. But then $\zeta\in(\mu+U)\cap T(\mu)$, a contradiction to (1). Therefore, we conclude that F is a generalized KKM mapping with respect to T.

Finally, since $T\in\mathrm{KKM}(X,X)$ and F is compact-valued, the family $\{F(x)\colon x\in X\}$ has the finite intersection property, so $\bigcap_{x\in X}F(x)\neq\emptyset$. Choosing $\eta\in\bigcap_{x\in X}F(x)$ and noting that $\bigcap_{x\in X}F(x)=K\setminus(\bigcup_{x\in X}(x+\frac{1}{2}U))$, we see that $\eta\notin\eta+\frac{1}{2}U$, a contradiction. Thus there is $x_U\in X$ such that $(x_U+U)\cap T(x_U)\neq\emptyset$. This completes the proof. □

For the remainder of this section, E will always denotes a locally convex topological vector space and X an almost convex subset of E.

Theorem 4.4. *If $T\in\mathrm{KKM}(X,X)$ is compact and closed, then T has a fixed point.*

Proof. Let $\{V_\alpha\colon \alpha\in\Lambda\}$ be a local base of 0 in E such that each V_α is convex. By Lemma 4.3, for each $\alpha\in\Lambda$ there is $x_\alpha\in X$ such that $(x_\alpha+V_\alpha)\cap T(x_\alpha)\neq\emptyset$. Choose $y_\alpha\in(x_\alpha+V_\alpha)\cap T(x_\alpha)$. Since $\{y_\alpha\}_{\alpha\in\Lambda}\subseteq\overline{T(X)}\subseteq X$ and $\overline{T(X)}$ is compact, we may assume that $y_\alpha\to y$ for some $y\in X$. Then we also have that $x_\alpha\to y$. It follows from the closeness of T that $y\in T(y)$. This completes the proof. □

The above theorem contains Lassonde [7, Theorem 4] as special case as the following corollary shows.

Corollary 4.5. *If $T: X \multimap X\in\mathcal{K}_c(X,X)$ is compact, then it has a fixed point.*

Proof. This follows immediately from Lemma 2.2(a), Corollary 3.13 and Theorem 4.4. □

Corollary 4.6. *If $T: X\multimap X$ is closed, compact, and convex-valued, then it has a fixed point.*

Proof. Since T is closed, compact, and convex-valued, it follows from Lemma 2.2(b) that T is a Kakutani mapping, and so the result follows from Corollary 4.5. □

Corollary 4.7. *Let Y be a compact subset of a topological vector space. If $T: X\multimap Y$ is convex-valued and closed, and $f\in\mathcal{C}(Y,X)$, then $f\circ T$ has a fixed point.*

Proof. Since T is closed and Y is compact, we see from Lemma 2.2(b) that T is u.s.c. and so, in conjunction with T being compact and convex-valued, $T \in \mathcal{K}(X, Y)$. Consequently, $f \circ T \in \mathcal{K}_c(X, X)$ and is compact. Hence $f \circ T$ has a fixed point. □

5. Applications

As applications of the fixed point theorems in the above section, we adopt the technique of Lassonde [7] to deduce some results related to von Neumann's Intersection Theorem. For a family of sets $\{X_i\}_{i\in I}$ and a fixed $i \in I$, we write X for $\prod_{j\in I} X_j$, and X^{-i} for the set $\prod_{j\in I, j\neq i} X_j$. If $x^{-i} \in X^{-i}$ and $j \in I$ with $j \neq i$, the jth coordinate of x^{-i} is denoted by x_j^{-i}. If $x_i \in X_i$ and $x^{-i} \in X^{-i}$, then $[x_i, x^{-i}]$ is the point of X defined as follows: its ith coordinate is x_i, and for the other jth coordinate is x_j^{-i}. Obviously, any $x \in X$ can be written as $x = [x_i, x^{-i}]$ for any $i \in I$, where x^{-i} denotes the projection of x onto X^{-i}. To any multifunction $G_i : X^{-i} \multimap X_i$, we associate the subset $\tilde{G}_i$ of X defined by $\tilde{G}_i = \{[x_i, x^{-i}]\colon x_i \in G_i(x^{-i})\}$.

Theorem 5.1. *Let X_0 be a nonempty almost convex subset of a topological vector space E_0, and X_i $(i = 1, \ldots, n)$ be a nonempty almost convex subset of a locally convex topological vector space E_i. For $i = 0, \ldots, n$, suppose $G_i \in \mathcal{K}_c(X^{-i}, X_i)$ and all the multifunctions G_i are compact except possibly G_n. Then $\bigcap_{i=0}^{n} \tilde{G}_i \neq \emptyset$.*

Proof. For $i \in \mathbb{Z}_{n+1}$, define $\Gamma_i : X^{-i} \multimap X^{-(i+1)}$ by

$$\Gamma_i(x^{-i}) = G_i(x^{-i}) \times \prod_{\substack{j\in\mathbb{Z}_{n+1}\\ j\notin\{i,i+1\}}} \{x_j^{-i}\} \quad \text{for } x^{-i} \in X^{-i}.$$

It is easy to see that $\Gamma_i \in \mathcal{K}_c(X^{-i}, X^{-(i+1)})$ for each $i \in \mathbb{Z}_{n+1}$. Therefore, the multifunction $\Gamma : X^{-0} \multimap X^{-0}$ defined by $\Gamma = \Gamma_n \Gamma_{n-1} \ldots \Gamma_0$ belongs to $\mathcal{K}_c(X^{-0}, X^{-0})$. Moreover, it is compact. Indeed, for $i = 0, \ldots, n-1$, since G_i is compact, there is a compact subset K_i of X_i such that $G_i(X^{-i}) \subseteq K_i \subseteq X_i$. So,

$$\Gamma_0(X^{-0}) \subseteq K_0 \times X_2 \times \cdots \times X_n,$$
$$\Gamma_1 \Gamma_0(X^{-0}) \subseteq K_0 \times K_1 \times X_3 \times \cdots \times X_n,$$

and, finally, $\Gamma_{n-1}\Gamma_{n-2} \ldots \Gamma_0(X^{-0}) \subseteq K_0 \times K_1 \times \cdots \times K_{n-1}$. Hence, $\Gamma(X^{-0})$ is contained in the compact set $\Gamma_n(K_0 \times K_1 \times \cdots \times K_{n-1})$, which shows that Γ is compact. By Proposition 4.2, X^{-0} is almost convex. So we can apply Corollary 4.5 to derive the existence of a point $x^{-0} \in X^{-0}$ such that $x^{-0} \in \Gamma(x^{-0})$. In other words, there exist $x^{-1} \in X^{-1}, \ldots, x^{-n} \in X^{-n}$ such that $x^{-(i+1)} \in \Gamma_i(x^{-i})$ for each $i \in \mathbb{Z}_{n+1}$, which means that

$$x_i^{-(i+1)} \in G_i(x^{-i}) \quad \text{for each } i \in \mathbb{Z}_{n+1} \tag{3}$$

and

$$x_j^{-(i+1)} = x_j^{-i} \quad \text{for each } j \in \mathbb{Z}_{n+1},\ j \notin \{i, i+1\}. \tag{4}$$

From (4), it follows that $x_j^{-i} = x_j^{-k}$ for any $i, j, k \in \mathbb{Z}_{n+1}$ with $j \notin \{i, k\}$. Hence, $[x_i^{-(i+1)}, x^{-i}] = [x_k^{-(k+1)}, x^{-k}]$ for any $i, k \in \mathbb{Z}_{n+1}$. Denote by x the point of X defined by $x = [x_i^{-(i+1)}, x^{-i}]$ for any $i \in \mathbb{Z}_{n+1}$. From (3), we derive that $x \in \tilde{G}_i$ for every $i \in \mathbb{Z}_{n+1}$. Hence $\bigcap_{i=0}^n \tilde{G}_i \neq \emptyset$. □

A generalized game is a game in which each player select a strategy in a subset determined by the strategies chosen by other players. Let $\mathbb{Z}_{n+1} = \{0, \dots, n\}$ denote the set of players, and for $i \in \mathbb{Z}_{n+1}$, let X_i denote the set of strategies of the ith player. Each element of $X = \prod_{i \in \mathbb{Z}_{n+1}} X_i$ determine an outcome. The payoff to the ith player is a real-valued continuous function f_i defined on X. Given $x^{-i} \in X^{-i}$ (the strategies of all others), the choice of the ith player is restricted to a nonempty compact subset $F_i(x^{-i})$ of X_i; the ith player chooses $x_i \in F_i(x^{-i})$ so as to maximize $f_i([x_i, x^{-i}])$. An equilibrium point in such a generalized game is a strategy vector $x \in X$ such that for all $i \in \mathbb{Z}_{n+1}$, $x_i \in F_i(x^{-i})$ and $f_i(x) = \max_{y_i \in F_i(x^{-i})} f_i([y_i, x^{-i}])$.

Theorem 5.2. *Let X_0 be a nonempty almost convex subset of a topological vector space E_0, and X_i $(i = 1, \dots, n)$ be a nonempty almost convex subset of a locally convex topological vector space E_i. For $i = 0, \dots, n$, let $F_i : X^{-i} \multimap X_i$ be a l.s.c. multifunction in $\mathcal{K}(X^{-i}, X_i)$ and let $f_i : X = \prod_{i=0}^n X_i \to \mathbb{R}$ be a continuous function such that for any fixed $x^{-i} \in X^{-i}$, the function $x_i \to f_i([x_i, x^{-i}])$ is quasi-concave on X_i. If all the multifunctions F_i are compact except possibly F_n, then there is an equilibrium point.*

Proof. For $i \in \mathbb{Z}_{n+1}$, define $G_i : X^{-i} \multimap X_i$ by

$$G_i\left(x^{-i}\right) = \left\{x_i \in F_i(x^{-i}):\ f_i\left([x_i, x^{-i}]\right) = \max_{y_i \in F_i(x^{-i})} f_i\left([y_i, x^{-i}]\right)\right\}.$$

Noting that an equilibrium point is a point of the intersection $\bigcap\{\tilde{G}_i:\ i \in \mathbb{Z}_{n+1}\}$, the theorem is proved if we show that the multifunctions G_i satisfy the assumptions of Theorem 5.1, that is, we have to show that each G_i is u.s.c. and each G_i is compact except possibly G_n. Let $i \in \mathbb{Z}_{n+1}$ be fixed. For any fixed $x^{-i} \in X^{-i}$, since the function $x_i \to f_i([x_i, x^{-i}])$ is continuous and quasi-concave on the nonempty compact convex set $F_i(x^{-i})$, the set $G_i(x^{-i})$ is nonempty, compact, and convex. Now define $T_i : X^{-i} \multimap X_i$ by

$$T_i\left(x^{-i}\right) = \left\{x_i \in X_i:\ f_i\left([x_i, x^{-i}]\right) \geqslant \max_{y_i \in F_i(x^{-i})} f_i\left([y_i, x^{-i}]\right)\right\}.$$

Obviously, $G_i = F_i \cap T_i$. Thus, if T_i is shown to be closed, then it follows from Lemma 2.2(b) that G_i is u.s.c. Since f_i is continuous on X, the functions $h, g : X \times X_i \to \mathbb{R}$ defined by $h(x, y_i) = f_i(x)$ and $g(x, y_i) = f_i(y_i, x^{-i})$ for each $(x, y_i) \in X \times X_i$ are continuous, and so the set

$$U_i = \left\{(x, y_i) \in X \times X_i:\ f_i(x) < f_i\left([y_i, x^{-i}]\right)\right\}$$

is open in $X \times X_i$. But,

$$\begin{aligned} X \setminus \tilde{T}_i &= \left\{x \in X:\ \text{there exists } y_i \in F_i(x^{-i}) \text{ such that } (x, y_i) \in U_i\right\} \\ &= \left\{x \in X:\ \left(\{x\} \times F_i(x^{-i})\right) \cap U_i \neq \emptyset\right\} \end{aligned}$$

is open in X because the multifunction $x \to \{x\} \times F_i(x^{-i})$ from X to $X \times X_i$ is l.s.c. Thus T_i is closed and hence G_i is u.s.c. Finally, noting that each $G_i(X^{-i})$ is contained in $F_i(X^{-i})$ and each F_i is compact except possibly F_n, we see that each G_i is compact except possibly G_n. This completes the proof. □

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